The never-ending mystery of π

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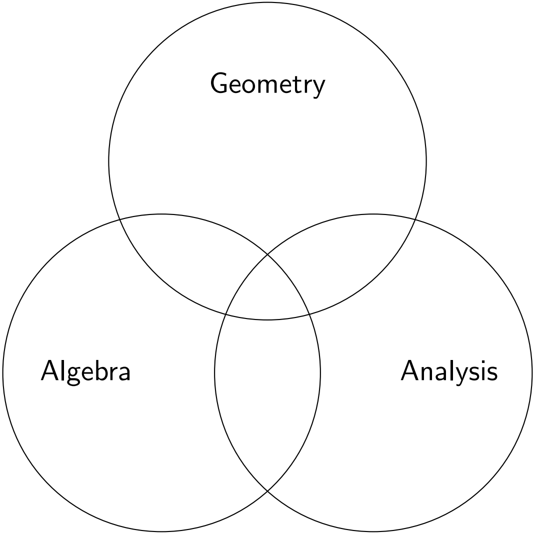
Tool boxes in mathematics



Also : Arithmetics, Combinatorics, Probability, Numerical simulation, ...

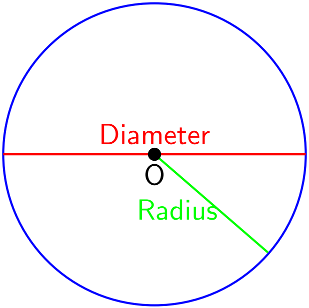
Outline

1. Geometry : shape of π
2. Numerics and Analysis : value of π and a magical formula
3. Algebra and Arithmetics : (ir)rationality and transcendence of π

What is π ?

Its origin : the shape of a circle !

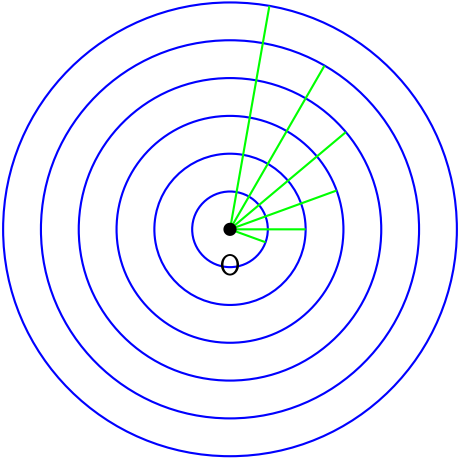
$$\pi = \frac{\text{Circumference of a circle}}{\text{Diameter of circle}} = \frac{\text{Circumference of a circle}}{2 \times \text{Radius of circle}}$$

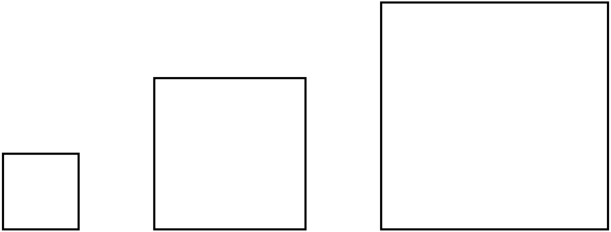


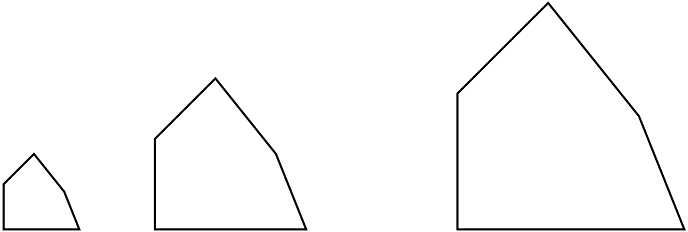
Question : Which circle ?

It doesn't matter ! The value of π is independent of the circle.

The ratio between the circumference and the radius is a constant ($=2\pi$).







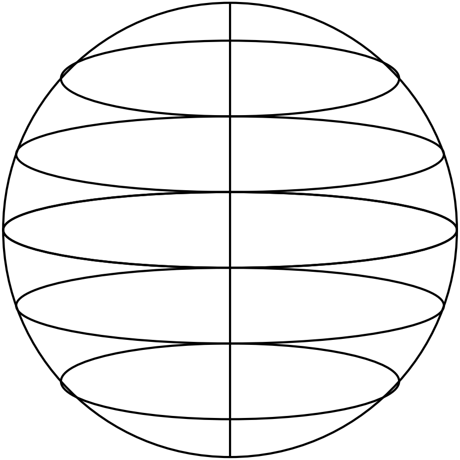
Transformations of the plane

- Translation : circumference stays invariant.
- Homothety : circumference is multiplied by the dilation factor.

This is true only on a flat surface, e.g. the Euclidien plane, the cylinder.

Challenge : Why it is not true on a non-flat surface, e.g. sphere ?
In mathematics, non-flatness is measured by curvature.

Gauss

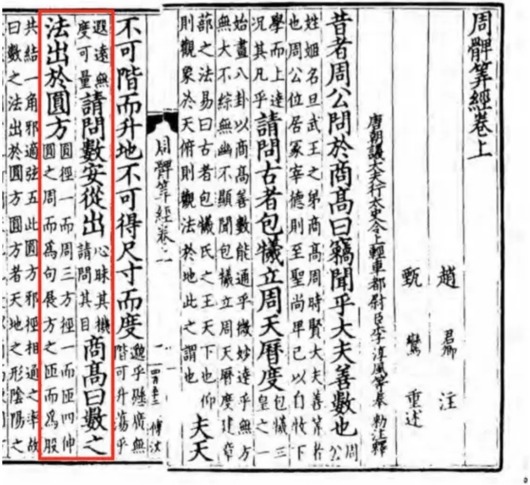


Gaussian curvature of a sphere $= \frac{1}{r^2}$
Question : What happens when $r \to \infty$?

To get a rough feeling of how big π is :

A picture from wikipedia

About 100 BC in China, $\pi \sim 3$



周髀算經卷上

趙君卿 注

甄鸞 重述

唐朝議大夫行太史令上輕車都尉臣李淳風等奉敕注釋

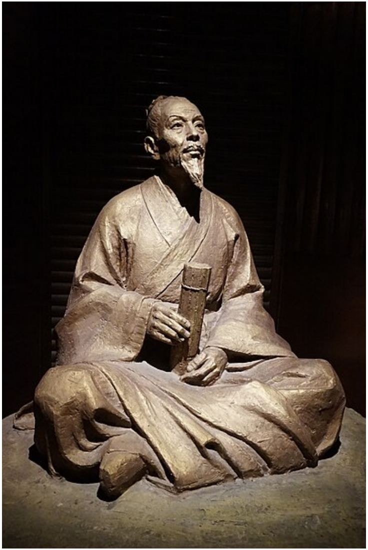
昔者周公問於商高曰竊聞乎大夫善數也周公姓姬名旦武王之弟商高周時賢大夫善筭者也周公位居冢宰德則至聖尚卑己以自牧下學而上達況其凡乎請問古者包犧立周天曆度包犧三皇之一始畫八卦以商高善數能通乎微妙達乎無方無大不綜無幽不顯聞包犧立周天曆度建章蔀之法易曰古者包犧氏之王天下也仰則觀象於天俯則觀法於地此之謂也夫天

周髀算經卷之一

不可階而升地不可得尺寸而度邈乎懸廣無階可升蕩乎遐遠無度可量請問數安從出心昧其機請問其目商高曰數之法出於圓方圓徑一而周三方徑一而匝四伸圓之周而為句展方之匝而為股共結一角邪適弦五此圓方邪徑相通之率故曰數之法出於圓方圓方者天地之形陰陽之

图 2 商高曰“数之法出于圆方”

About 480 AD in China, $\pi \sim 3.1415926$



Zu Chongzhi

Approximate value of π : in decimals

3.1415926535 8979323846 2643383279 5028841971 6939937510
5820974944 5923078164 0628620899 8628034825 3421170679
8214808651 3282306647 0938446095 5058223172 5359408128
4811174502 8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165 2712019091
4564856692 3460348610 4543266482 1339360726 0249141273
7245870066 0631558817 4881520920 9628292540 9171536436
7892590360 0113305305 4882046652 1384146951 9415116094
3305727036 5759591953 0921861173 8193261179 3105118548
0744623799 6274956735 1885752724 8912279381 8301194912
9833673362 4406566430 8602139494 6395224737 1907021798
6094370277 0539217176 2931767523 8467481846 7669405132
0005681271 4526356082 7785771342 7577896091 7363717872
1468440901 2249534301 4654958537 1050792279 6892589235
4201995611 2129021960 8640344181 5981362977 4771309960
5187072113 4999999837 2978049951 0597317328 1609631859
5024459455 3469083026 4252230825 3344685035 2619311881
7101000313 7838752886 5875332083 8142061717 7669147303
5982534904 2875546873 1159562863 8823537875 9375195778
1857780532 1712268066 1300192787 6611195909 2164201989
3809525720 1065485863 8823537875 9375195778 1857780532
1712268066 1300192787 6611195909 2164201989 3809525720
1065485863 8823537875 9375195778 1857780532 1712268066
1300192787 6611195909 2164201989 3809525720 1065485......

Basel problem

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = ?$$

A question posed by Pietro Mengoli in 1650, answered by Leonhard Euler in 1735, and named after Basel (hometown of Euler).

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{5^2} = 1.46361111111111...$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{10^2} = 1.54976773116654...$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{100^2} = 1.634983900184892..$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{1000^2} = 1.64393456668155...$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{10000^2} = 1.64483407184805...$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{100000^2} = 1.64492406689822...$$

$$\vdots$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = {\color{red}1.644934066848226...}$$

The guess

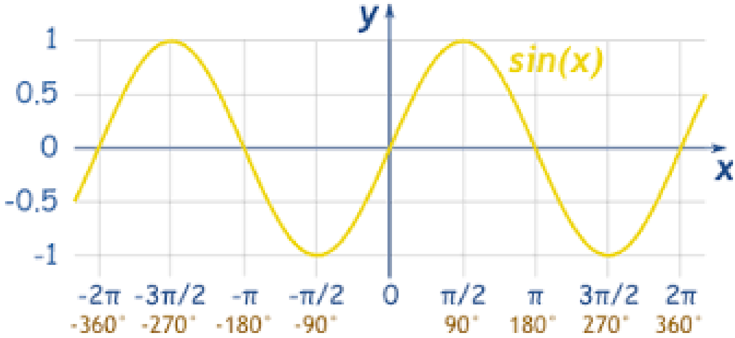
$$\sqrt{1.644934066848226\ldots \times 6}$$

$$=3.14159265358979323846264338327950288419716939937510$$
$$58209749445923078164062862089986280348253421170679 82$$
$$1480865132823066470938446\ldots$$

Conjecture

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \stackrel{?}{=} \frac{\pi^2}{6}$$

Sine function



Reproducing a "polynomial" from its zeros :

$$\sin(x) = x(1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi}) \cdots$$

$$\sin(x)/x = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right) x^2 + \cdots$$

Approximating $\sin(x)/x$ by polynomial :

$$\sin(x)/x = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Conclusion :

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Riemann zeta function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

Riemann

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$
$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$
$$\zeta(6) = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots = \frac{\pi^6}{945}$$
$$\zeta(8) = \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \cdots = \frac{\pi^8}{9450}$$

...

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n} \cdot (2\pi)^{2n}}{2 \cdot (2n)!}.$$

Here B_{2n} is an easily computable rational number (Bernoulli number).

Approximate value of π : in decimals

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Natural questions

- Does it end ?
- Is there a pattern ?
- Does it repeat after a while ?

Example

$\frac{1}{2} = 0.5$
$\frac{3}{125} = 0.024$
$\frac{1}{7} = 0.142857142857142857... = 0.\overline{142857}$
$\frac{8}{65} = 0.2153846153846153846... = 0.2\overline{153846}$

What does it mean for a number to be rational ?

Fact : a number is rational if and only if its decimal representation is eventually repeating.

Question : Is π rational, that is, the ratio of two integers ?

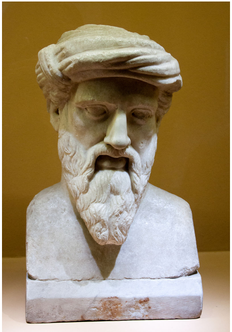
Euclid

Archimedes

- $\frac{22}{7} \sim 3.14\ 2857...$
- Relative Error : 0.04%

- $\frac{355}{113} \sim 3.141592\ 9203...$
- Relative Error : 0.000264%

$\sqrt{2}$ is not rational.



Pythagoras

Hippasus

Exercise : $\log_2 3$ is not rational.

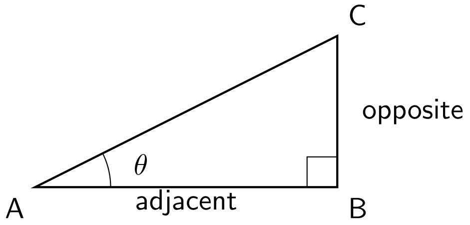
Lambert's proof of irrationality of π

Lambert

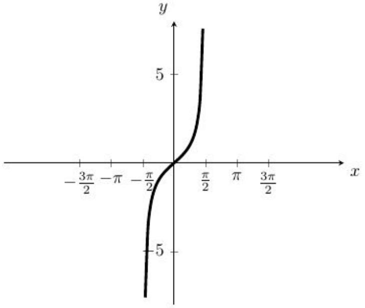
Some trigonometry

The tangent of an angle θ is defined as :

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$



Key fact : tan(non-zero rational number) is *never* a rational number !



Graph of $y = \tan(x)$

We know : $\tan(\frac{\pi}{4}) = 1$.
Conclusion : π is not a rational numbe.

Proof of the Key Fact

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \ddots}}}}}$$

Use two formulas :

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Proof of the Key Fact

Assume, for contradiction, that $x = \frac{p}{q}$ and $\tan(x) = \frac{a}{b}$:

$$\frac{a}{b} = \cfrac{\frac{p}{q}}{1 - \cfrac{\left(\frac{p}{q}\right)^2}{3 - \cfrac{\left(\frac{p}{q}\right)^2}{5 - \cfrac{\left(\frac{p}{q}\right)^2}{7 - \ddots}}}} = \cfrac{p}{q - \cfrac{p^2}{3q - \cfrac{p^2}{5q - \cfrac{p^2}{7q - \ddots}}}}$$

We get an infinite descending chain of positive integers !

See the video here by for an excellent explanation of Lambert's proof.

Further results

Theorem. $\zeta(2)$ is not a rational number. Hence π^2 is not a raitonal number.

Stronger result :
Theorem. $\zeta(2n)$ is not a rational number. Hence π^{2n} is not a raitonal number.

In fact, much much stronger result :
Theorem (Lindemann–Weierstrass). π is transcendental, that is, π is NOT a root of any polynomial with coefficients in $\mathbb{Q}$.
Example : $\sqrt{2}$ is not rational but not transcendental.

Questions and Challenges

- Is $\zeta(3)$ rational ? Look for Apéry online.
- are $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$, etc rational ? Are they transcendental ?
- Is π^π rational ?
- Is $e + \pi$ rational or even transcendental ?

Thank you !